AIAA JOURNAL
Vol. 36, No. 4, April 1998

Comparison of Various Orthogonal Polynomials in *hp*-Version Time Finite Element Method

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Four types of orthogonal polynomials—Legendre, Chebyshev, Hermite, and integrated Legendre—are evaluated as basis functions in the time finite element method to solve initial value problems governed by second-order differential equations. Condition numbers of the augmented stiffness matrix for the selected problems are estimated by increasing the number of polynomial terms in the expansion. Results for the CPU time using an IBM 3090-300E/VF computer and the estimated condition numbers, using IMSL subroutine DLFCRG, for each of four basis functions are presented. The augmented stiffness matrix for the case of integrated Legendre polynomials is found to have the best condition number behavior, and that for the Hermite polynomial the worst. The integrated Legendre polynomials require the most CPU time, whereas the Chebyshev polynomials require the least.

Introduction

In recent years, the time finite element method has emerged as a viable approach for studying the transient response of systems, i.e., for solving initial value problems. Most works attempted to use Hamilton's principle as a starting point for the finite element formulation of initial value problems. In search for an alternative formulation for finding the response and sensitivity of transient problems, Kapania and Park,[1,2] in a recent study, extended the bilinear formulation suggested by Peters and Izadpanah[3] for linear undamped systems to linear and nonlinear, damped and undamped systems. The bilinear formulation in the time domain benefits from the large foundation of mathematical theorems[4] and knowledge already developed for the p-version finite element method.[5] It has been implemented to study various aspects of stress analysis and has shown very good results, including problems with steep gradients. In the p-version, the domain of interest is divided into convex subdomain, and the polynomial approximants are piecewise smooth only over individual convex subdomains. This gives a greater versatility and higher rate of convergence.

However, most previous works were concentrated on theoretical aspects with less emphasis on computational ones. Rossow and Katz[6] demonstrated the computational efficiency obtainable using hierarchical nodal variables and precomputed arrays. They obtained computation times for the element stiffness matrix for several different problem types and polynomial orders, and the results were compared with previously available results. Carnevali et al.[7] showed the sparsity patterns and conditioning of the stiffness matrix by using several types of the new basis functions in three-dimensional linear elastostatics and dynamics. For providing better sparsity and conditioning properties, they used the hierarchical basis functions for the p-version finite element, which satisfy the completeness and convergence as the polynomial order increased.

For initial value problems, the p-version time finite element method can also be used for solving multi-degree-of-freedom problems and mixed initial value and boundary value problems. The method yields an excellent approximation by using a relatively large time step size. Problems treated by Kapania and Park[2] were the transient response and the response sensitivity of the van der Pol oscillator, mass on a hardening and a softening spring system, and a two-degree-of-freedom system having cubic nonlinearities. For all of the cases studied, the results were obtained using Legendre polynomials as basis functions. The results were also obtained using other polynomials, namely, Chebyshev, Hermite, and integrated Legendre polynomials as basis functions but without any significant difference in the accuracy. The objective of this study is to compare the performance of these polynomials in terms of the condition number of the resulting matrices and the CPU time required for solving the resulting set of nonlinear equations.

Bilinear Formulations

The differential equation describing a nonlinear oscillatory system over a given length of time, $T_0 < t \le T_f$, may have a general form:

$$\boldsymbol{g}(u,\ \dot{u},\ \ddot{u},\ t,\ \boldsymbol{x}) = 0 \tag{1}$$

where $\boldsymbol{g}$ may be a nonlinear function of u and $\dot{u}$. Also u, $\dot{u}$, and $\ddot{u}$ are functions of time t. The vector x_k denotes design parameters of the system. The bilinear formulation[1–3] of Eq. (1) yields an expression of the form

$$\tilde{\boldsymbol{g}}(\boldsymbol{q},\ \boldsymbol{x}) = 0 \tag{2}$$

where the vector q_j denotes the generalized coordinates.

Equation (2), at times, may be written as

$$\tilde{\boldsymbol{g}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{q} = 0 \tag{3}$$

where A_i is the load vector and B_{ij} is the nonlinear stiffness matrix and a function of generalized coordinates. Furthermore, the sensitivity of the transient response of a nonlinear system can be obtained by taking the derivative of Eq. (3) with respect to design parameter x_k.

The result may be expressed as

$$\frac{\partial A_i}{\partial x_k} - \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial x_k} q_j - \sum_{m=1}^{N} \left(\sum_{j=1}^{N} B_{ij} \frac{\partial q_j}{\partial q_m} + \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial q_m} q_j \right) \frac{\partial q_m}{\partial x_k} = 0 \tag{4}$$

This reduces symbolically, in the matrix form, as

$$\left[B_{ij}^{*}\right]\left\{\frac{\partial q_j}{\partial x_k}\right\} + \left[\frac{\partial B_{ij}}{\partial x_k}\right]\{q_j\} = \left\{\frac{\partial a_i}{\partial x_k}\right\} \tag{5}$$

where

$$B_{ij}^{*} = B_{ij} + \sum_{m=1}^{N} \frac{\partial B_{im}}{\partial q_j} q_m \tag{6}$$

As shown earlier, the nonlinear stiffness matrices $\boldsymbol{B}$ and $\boldsymbol{B}^{*}$ are not generally equal in the nonlinear system.

Received March 10, 1997; revision received Oct. 15, 1997; accepted for publication Nov. 8, 1997.

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By adding an initial condition and imposing a constraint in the form of Lagrange multipliers, the augmented stiffness matrix $\bar{\boldsymbol{B}}$ for the transient response may be expressed in matrix form as

$$\bar{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} & \{\psi_i(T_f)\} \\ \langle \chi_j(T_0) \rangle & 0 \end{bmatrix} \tag{7}$$

whereas the augmented stiffness matrix $\bar{\boldsymbol{B}}^*$ for the response sensitivity may be expressed as

$$\bar{\boldsymbol{B}}^* = \begin{bmatrix} \boldsymbol{B}^* & \{\psi_i(T_f)\} \\ \langle \chi_j(T_0) \rangle & 0 \end{bmatrix} \tag{8}$$

where $\psi_i(t)$ and $\chi_j(t)$ are polynomial basis functions for test and trial functions, respectively. In Eq. (8), { } denotes the column vector and ⟨ ⟩ denotes the row vector. In the case of two-degree-of-freedom systems, the augmented stiffness matrix $\bar{B}_{ij}$, derived for the first time in Ref. 8, for the transient response may be expressed as

$$\bar{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}^{(1)} & 0 & \{\psi_i(T_f)\} & 0 \\ 0 & \boldsymbol{B}^{(2)} & 0 & \{\psi_i(T_f)\} \\ \langle \chi_j(T_0) \rangle & 0 & 0 & 0 \\ 0 & \langle \chi_j(T_0) \rangle & 0 & 0 \end{bmatrix} \tag{9}$$

whereas the augmented stiffness matrix $\bar{B}^*_{ij}$ for the response sensitivity takes the form

$$\bar{\boldsymbol{B}}^* = \begin{bmatrix} \boldsymbol{B}^{*(11)} & \boldsymbol{B}^{*(12)} & \{\psi_i(T_f)\} & 0 \\ \boldsymbol{B}^{*(21)} & \boldsymbol{B}^{*(22)} & 0 & \{\psi_i(T_f)\} \\ \langle \chi_j(T_0) \rangle & 0 & 0 & 0 \\ 0 & \langle \chi_j(T_0) \rangle & 0 & 0 \end{bmatrix} \tag{10}$$

The detailed derivation of the various elements in Eqs. (8–10) for the two-degree-of-freedom system considered here is provided in Ref. 8.

Basis Functions

Because one goal of the computational method is the computational simplicity, one choice for the basis function is polynomials. Basically the orthogonal polynomials[9] are selected to have simplicity in the computation and well behavedness of the stiffness matrix. The orthogonality has less effect on the computation due to the nature of the formulation. Four sets of orthogonal polynomials—Legendre, Chebyshev, Hermite, and integrated Legendre—are selected as basis functions, each defined over the range $-1 < t \leq 1$. In integrated Legendre polynomials, the first and second terms are linear combinations of the first two integrals of Legendre polynomials:

$$I_1(t) = (1 - t)/2, \qquad I_2(t) = (1 + t)/2 \tag{11}$$

The rest of the polynomials $I_n(t)$ are found using the following relationship[10]:

$$I_n(t) = \frac{1}{\sqrt{2(2n-3)}}[P_{n-1}(t) - P_{n-3}(t)], \qquad n \geq 3 \tag{12}$$

where $P_n(t)$ denotes Legendre polynomials.

Numerical Results

Condition numbers of the augmented stiffness matrix for the selected problems are estimated by increasing the number of polynomial terms in the expansion. Results for the CPU time and the estimated condition numbers for each of four basis functions are presented. Condition numbers for the response sensitivity are almost the same as those for the transient response case, and so the results are shown only for the transient response case. The CPU time for the selected problems required only 1–2 s (except in the case of van der Pol's oscillator). Those results are not presented here. To estimate the condition numbers, the IMSL subroutine DLFCRG was used. All the computations were performed on an IBM 3090-300E/VF running under VM/CMS operating systems. The detailed derivations of the used nonlinear stiffness matrices for the three single-degree-of-freedom system examples are given in Ref. 1 and those for the two-degree-of-freedom system in Ref. 8.

van der Pol's Oscillator

Consider a nonlinear van der Pol's equation[11] with a large parameter ϵ given by

$$\ddot{u}(t) + \epsilon[u(t)^2 - 1]\dot{u}(t) + u(t) = 0$$

$$\epsilon = 5.0, \qquad 0 < t \leq 20 \tag{13}$$

with initial conditions

$$u(0) = 2.0, \qquad \dot{u}(0) = 0.0$$

The response of van der Pol's equation for a large parameter ϵ shows a slow buildup followed by a sudden discharge, repeated periodically. Condition numbers of the augmented stiffness matrix were estimated by varying the order of polynomials from 3 to 8 with fixed time steps of $\Delta t = 0.005$. Results of the estimated condition numbers and the CPU time are presented in Fig. 1 and Table 1, respectively. From the results shown in Fig. 1, condition numbers are generally increasing along with the order of polynomials, except for the case of integrated Legendre polynomials. For the latter case, they are independent of the polynomial order.

Condition numbers obtained using Hermite polynomials of degrees 6–8 show extremely high values compared with the condition numbers obtained for other polynomials. This is because the numerical values of diagonal terms are getting larger whenever a polynomial term is added in the expansion for the basis function (see Appendix B). The condition numbers for the case of integrated Legendre polynomials consistently have far smaller values. This

Table 1 CPU time for calculating transient response and response sensitivity: van der Pol's oscillator

N^a	Legendre	Chebyshev	Hermite	Integrated Legendre
3	6.0 (9.0)	5.0 (8.0)	6.0 (8.0)	7.0 (12.0)
4	6.0 (12.0)	6.0 (11.0)	6.0 (11.0)	8.0 (17.0)
5	7.0 (17.0)	7.0 (14.0)	7.0 (15.0)	9.0 (24.0)
6	9.0 (23.0)	8.0 (19.0)	8.0 (21.0)	12.0 (33.0)
7	11.0 (30.0)	9.0 (25.0)	10.0 (27.0)	14.0 (44.0)
8	13.0 (39.0)	11.0 (32.0)	12.0 (36.0)	17.0 (57.0)

[a]Order of polynomials.

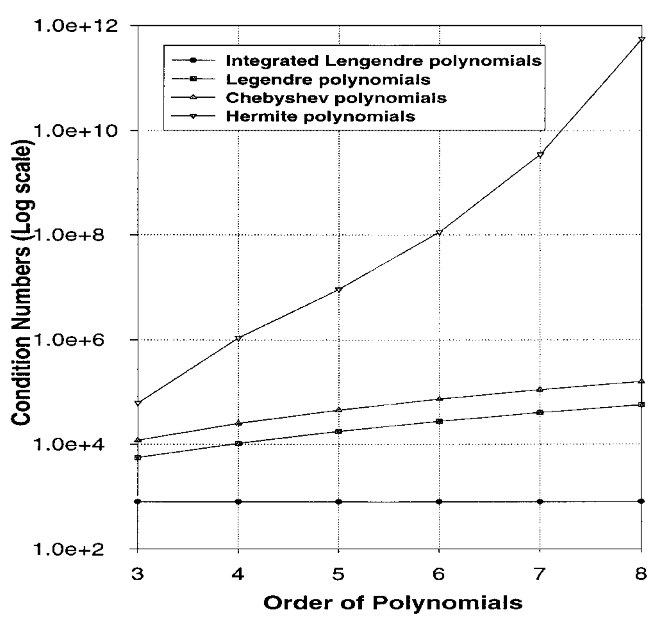

Fig. 1 Condition numbers of augmented stiffness matrix: van der Pol's oscillator.

is because the numerical values of diagonal terms in the stiffness matrix remained the same even after polynomial terms were added to the basis function expansion (see Appendix B). Condition numbers obtained using Legendre and Chebyshev polynomials show a similar behavior but with a little higher values for the Legendre polynomials. Note that the augmented stiffness matrix using Chevyshev polynomials is not sparse as is the case for Legendre polynomials. This is due to nonuniform weights being used for Chevyshev polynomials.

Table 1 presents the required CPU time for calculating the transient response and the response sensitivity (in parentheses) of the given system. Note that the computation time is increasing as the order of polynomials increases. The integrated Legendre polynomials require more computation time as compared with the other polynomials. This is expected because an integration process is needed for generating integrated Legendre polynomials, whereas others are generated by appropriate recurrence procedures.

Mass on a Nonlinear Hardening Spring

Consider a damped oscillation of a mass on a nonlinear hardening spring over a given length of time, $0 < t \le 5$, which is defined by

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) + \mu u(t)^3 = 0.0 \tag{14}$$

with initial conditions

$$u(0) = 1.0, \qquad \dot{u}(0) = 0.0$$

for the following values of system parameters:

$$M = 1.0, \qquad C = 4.0, \qquad K = 400.0, \qquad \mu = 1.0$$

The domain, $0 < t \le 5$, is divided into 50 elements of equal time steps. Condition numbers were estimated by varying the order of polynomials from 3 to 8. Results for the estimated condition numbers are shown in Fig. 2.

We failed to estimate condition numbers for Hermite polynomials of degrees 6–8. Again the integrated Legendre case shows smallest condition numbers, and the condition number is independent of the order of the polynomial being used.

Mass on a Nonlinear Softening Spring

A damped oscillation of a mass on a nonlinear softening spring model with a time length, $0 < t \le 5$, is represented by

$$M\ddot{u}(t) + C\dot{u}(t) + \alpha \tanh[u(t)] = 0 \tag{15}$$

with initial conditions

$$u(0) = 0, \qquad \dot{u}(0) = 25.0$$

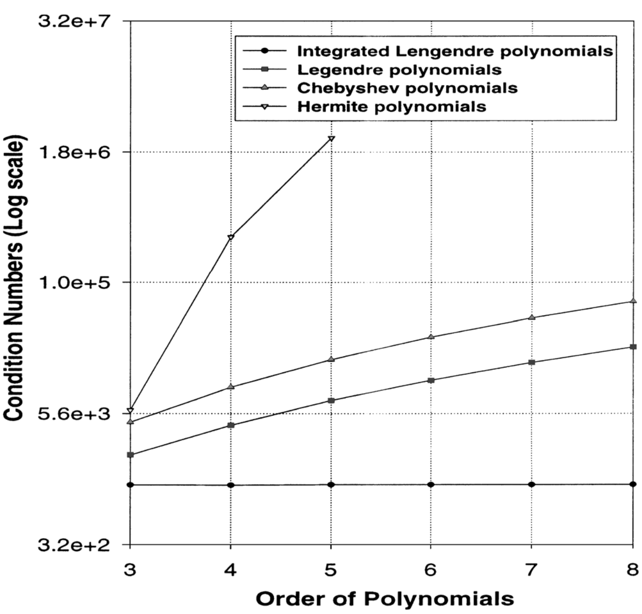

Fig. 2 Condition numbers of augmented stiffness matrix: hardening spring model.

for the following values of system parameters:

$$M = 1.0, \qquad C = 1.0, \qquad \alpha = 100.0$$

To estimate condition numbers, the domain was divided into 50 elements of equal time steps, and the order of polynomials were again increased from 3 to 8. Figure 3 shows condition numbers of the augmented stiffness matrix.

We noted that the integrated Legendre case again shows smallest values and remains unchanged as the number of terms is increased. The Hermite case shows extremely high condition numbers as the order of the polynomial is increased.

Two-Degree-of-Freedom System Having Cubic Nonlinearities

As a fourth example, a two-degree-of-freedom system governed by differential equations having cubic nonlinearities[12] is considered:

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2 + \alpha_3 u_1 u_2^2 + \alpha_4 u_2^3 = 0 \tag{16}$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \alpha_5 u_1^3 + \alpha_6 u_1^2 u_2 + \alpha_7 u_1 u_2^2 + \alpha_8 u_2^3 = 0 \tag{17}$$

with initial conditions

$$u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0$$

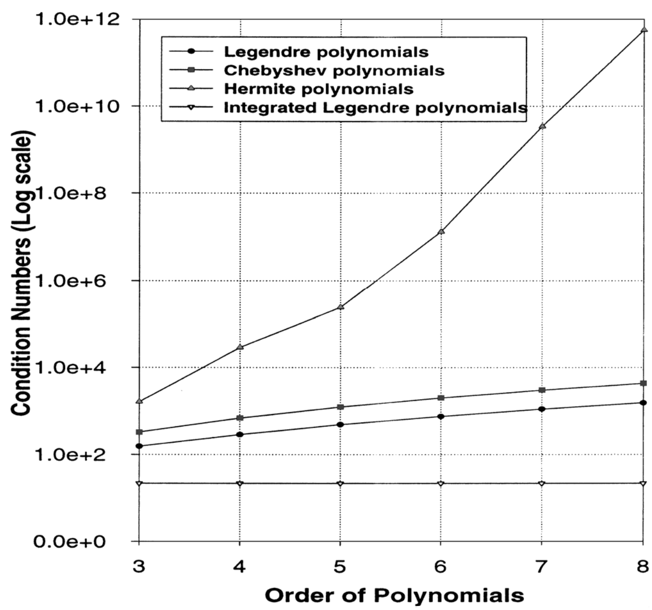

Fig. 3 Condition numbers of augmented stiffness matrix: softening spring model.

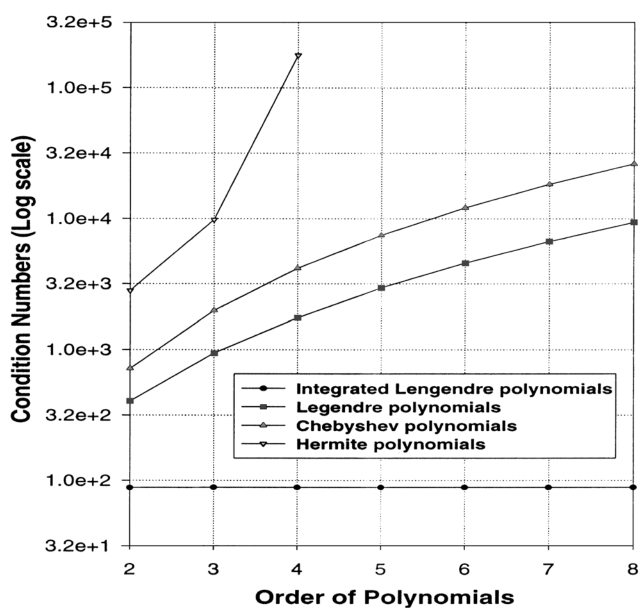

Fig. 4 Condition numbers of augmented stiffness matrix: two-degree-of-freedom system with cubic nonlinearities.

for the following values of the system parameters:

$$\omega_1^2 = 25.0, \qquad \mu_1 = 0.35$$
$$\alpha_1 = 5.0, \qquad \alpha_2 = 0.5, \qquad \alpha_3 = 0.25, \qquad \alpha_4 = 3.0$$
$$\omega_2^2 = 17.0, \qquad \mu_2 = 0.25$$
$$\alpha_5 = 2.5, \qquad \alpha_6 = 0.75, \qquad \alpha_7 = 0.2, \qquad \alpha_8 = 5.0$$

The domain with a range of time, $0 < t \leq 10$, is divided into 100 elements of equal time steps. The estimated condition numbers of the augmented stiffness matrix is shown in Fig. 4. The results show very similar trends as those for the case of single-degree-of-freedom systems.

In the case of Hermite polynomials, again we failed to estimate the condition numbers when higher-order polynomials (5–8) were used. The integrated Legendre case shows the lowest numbers among all of the polynomials.

Conclusion

Four cases of nonlinear second-order systems are examined for finding the best polynomials to be used as the basis function in the hp-version of the time finite element method of solving initial value problems. Four different types of polynomials, namely, Hermite, Chebyshev, Legendre, and integrated Legendre, are evaluated. Condition numbers of the resulting stiffness matrices are obtained as a function of number of terms used in the expansion. For all four examples, the integrated Legendre polynomials show the best performance. The choice of Hermite polynomials is not recommended based on the results presented here. Compared with the other polynomials, integrated Legendre polynomials require large CPU times.

Appendix A: Stiffness Matrices for Single-Degree-of-Freedom Systems

van der Pol's Oscillator

The nonlinear stiffness matrices, $\boldsymbol{B}$ and $\boldsymbol{B}^*$, for van der Pol's oscillator take the following forms:

$$B_{ij} = \int_{T_0}^{T_f} \left\{ \psi_i \chi_j + \epsilon \psi_i \dot{\chi}_j \left[\left(\sum_{l=1}^{N} q_l \chi_l \right)^2 - 1 \right] - \dot{\psi}_i \dot{\chi}_j \right\} \mathrm{d}t \qquad \text{(A1)}$$

$$B_{ij}^* = B_{ij} + \int_{T_0}^{T_f} 2\epsilon \psi_i \chi_j \left(\sum_{l=1}^{N} q_l \chi_l \right) \left(\sum_{m=1}^{N} q_m \dot{\chi}_m \right) \mathrm{d}t \qquad \text{(A2)}$$

where B_{ij}^* is explicitly derived from Eq. (6).

Mass on a Nonlinear Hardening Spring

The stiffness matrix B_{ij} for the transient response has the following form:

$$B_{ij} = \int_{T_0}^{T_f} (K \psi_i \chi_j + C \psi_i \dot{\chi}_j - M \dot{\psi}_i \dot{\chi}_j)\, \mathrm{d}t \qquad \text{(A3)}$$

whereas the nonlinear stiffness matrix B_{ij}^* for the response sensitivity takes the following form:

$$B_{ij}^* = B_{ij} + \int_{T_0}^{T_f} 3\mu \psi_i \chi_j \left(\sum_{l=1}^{N} q_l \chi_l \right)^2 \mathrm{d}t \qquad \text{(A4)}$$

Mass on a Nonlinear Softening Spring

The stiffness matrix B_{ij} for the transient response takes the form

$$B_{ij} = \int_{T_0}^{T_f} (C \psi_i \dot{\chi}_j - M \dot{\psi}_i \dot{\chi}_j)\, \mathrm{d}t \qquad \text{(A5)}$$

whereas the nonlinear stiffness matrix B_{ij}^* for the response sensitivity takes the form

$$B_{ij}^* = B_{ij} + \int_{T_0}^{T_f} \frac{K \psi_i \chi_j}{\cosh^2\left(\sum_{l=1}^{N} q_l \chi_l\right)}\, \mathrm{d}t \qquad \text{(A6)}$$

Appendix B: Augmented Stiffness Matrix for van der Pol's Oscillator

Numerical values of the augmented matrix for the transient response case in van der Pol's oscillator are presented. The values were generated using Hermite and integrated Legendre polynomials of degree 7 as basis functions. The values for the response sensitivity case are similar.

Two-digit rounding values of the matrix $\boldsymbol{B}$ using Hermite polynomials are reported as

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{-0.50E-2} & 0.31E+2 & -0.33E-2 & -0.63E+2 & -0.40E-2 & -0.13E+3 & 0.10E+1 \\ 0.00E+0 & \mathbf{-0.32E+4} & 0.84E+2 & 0.64E+4 & -0.60E+3 & 0.13E+5 & 0.20E+1 \\ -0.33E-2 & -0.21E+2 & \mathbf{-0.17E+5} & 0.18E+3 & 0.12E+6 & -0.18E+4 & 0.20E+1 \\ 0.00E+0 & 0.64E+4 & -0.30E+3 & \mathbf{-0.54E+5} & 0.24E+4 & 0.56E+6 & -0.40E+1 \\ -0.40E-2 & -0.25E+2 & 0.12E+6 & -0.11E+4 & \mathbf{-0.96E+6} & 0.17E+5 & -0.20E+2 \\ 0.00E+0 & 0.13E+5 & 0.16E+4 & 0.56E+6 & -0.14E+5 & \mathbf{-0.85E+7} & -0.80E+1 \\ 0.10E+1 & -0.20E+1 & 0.20E+1 & 0.40E+1 & -0.20E+2 & 0.80E+1 & 0.00E+0 \end{bmatrix}$$

Two-digit rounding values of the matrix $\boldsymbol{B}$ using integrated Legendre polynomials are reported as

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{-0.20E+3} & 0.20E+3 & 0.10E+1 & 0.33E+0 & -0.16E+0 & 0.00E+0 & 0.00E+0 \\ 0.20E+3 & \mathbf{-0.20E+3} & -0.11E+1 & 0.31E+0 & 0.16E+0 & 0.00E+0 & 0.10E+1 \\ -0.13E+1 & 0.13E+1 & \mathbf{-0.40E+3} & -0.96E+0 & -0.49E-2 & 0.10E+1 & 0.00E+0 \\ -0.36E-2 & 0.36E-2 & 0.74E+0 & \mathbf{-0.40E+3} & -0.56E+0 & -0.32E-2 & 0.00E+0 \\ 0.54E-1 & -0.54E-1 & 0.31E-2 & 0.51E+0 & \mathbf{-0.40E+3} & -0.41E+0 & 0.00E+0 \\ 0.00E+0 & 0.00E+0 & -0.50E-1 & 0.23E-2 & 0.38E+0 & \mathbf{-0.40E+3} & 0.00E+0 \\ 0.10E+1 & 0.00E+0 & 0.00E+0 & 0.00E+0 & 0.00E+0 & 0.00E+0 & 0.00E+0 \end{bmatrix}$$

Acknowledgment

Portions of this paper were first presented at the 1995 SIAM Annual Meeting, Charlotte, North Carolina, Oct. 23–26, 1995.

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